Homework 2

A. Chorin

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Homework 2, due Wednesday Feb. 9.

1. Write a Monte-Carlo program to evaluate $\int_0^1 \frac{e^{-\sqrt{1-x^2}}}{\sqrt{x}} dx$ (you may have to do importance sampling to get anything at all). As you sum the samples estimate the variance and the error; by making several runs with different sample sizes check that you error estimates are realistic. Estimate the number of samples needed to get an error $\leq \epsilon$, where $\epsilon > 0$ is a tolerance. Find the value of the integral with an error ≤ 1 per cent.

2. Let $H_0, H_1, H_2, ...$ be Hermite polynomials (H_n is a polynomial of degree n, $\int_{-\infty}^{+\infty} H_m H_n e^{-x^2}/\sqrt{\pi} dx = 0$ if $n \neq m$, $= 1$ if $n = m$). Suppose you want to evaluate $I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} g(x) e^{-x^2} dx$, where g is a given function; let ξ be a Gaussian variable with mean 0 and variance 1/2. Show that for all b, c, $I = E[g(\xi) + bH_1(\xi) + cH_2(\xi)]$. However, the variance of the estimator is not independent of b, c. What values should a, b take to yield an estimator of least variance?

3. Let η be a random variable which takes the value 1/2 with probability 1/2 and the value $-1/2$ also with probability 1/2. Let $\Xi_n = (\sum_1^n \eta_i)/n$, where the η_i are independent variables with the same distribution as η. Find the values that Ξ_n can take and their probabilities for $n = 3, 6, 9$, and plot their histograms together with the pdf of the limit of Ξ_n as $n \to \infty$.

4. Check the correctness of the derivation of Box-Muller sampling scheme (note that there are typos in the class notes).

5. An exponential variable with parameter λ has the density $f = \lambda e^{-\lambda x}, \lambda > 0$. If you are given n independent samples of such a variable, how do you find the maximum likelihood estimate of λ?